Guts Round

Lexington High School

March 23, 2019

10th Annual Lexington Math Tournament - Guts Round - Part 1

Team Name: ______________________

______________ 1. **[5]** Alice has a pizza with eight slices. On each slice, she either adds only salt, only pepper, or leaves it plain. Determine how many ways there are for Alice to season her entire pizza.

______________ 2. **[5]** Call a number *almost prime* if it has exactly three divisors. Find the number of *almost prime* numbers less than 100.

______________ 3. **[5]** Determine the maximum number of points of intersection between a circle and a regular pentagon.

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______________ 4. **[5]** Let $d(n)$ denote the number of positive integer divisors of n. Find $d(d(20^{18}))$.

______________ 5. **[5]** 20 chubbles are equal to 19 flubbles. 20 flubbles are equal to 18 bubbles. How many bubbles are 1000 chubbles worth?

______________ 6. **[5]** Square $ABCD$ and equilateral triangle EFG have equal area. Compute $\frac{AB}{EF}$.

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______________ 7. **[6]** Find the minimum value of y such that $y = x^2 - 6x - 9$ where x is a real number.

______________ 8. **[6]** I have 2 pairs of red socks, 5 pairs of white socks, and 7 pairs of blue socks. If I randomly pull out one sock at a time without replacement, how many socks do I need to draw to guarantee that I have drawn 3 pairs of socks of the same color?

______________ 9. **[6]** There are 23 paths from my house to the school, 29 paths from the school to the library, and 3 paths from the library to town center. Additionally, there are 6 paths directly from my house to the library. If I have to pass through the library to get to town center, how many ways are there to travel from my house all the way to the town center?

10th Annual Lexington Math Tournament - Guts Round - Part 4

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______________ 10. **[6]** A circle of radius 25 and a circle of radius 4 are externally tangent. A line is tangent to the circle of radius 25 at A and the circle of radius 4 at B, where $A \neq B$. Compute the length of AB.

______________ 11. **[6]** A gambler spins two wheels, one numbered 1 to 4 and another numbered 1 to 5, and the amount of money he wins is the sum of the two numbers he spins in dollars. Determine the expected amount of money he will win.

______________ 12. **[6]** Find the remainder when 20^{19} is divided by 18.

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______________ 13. **[7]** Two concentric circles have radii 1 and 3. Compute the length of the longest straight line segment that can be drawn from a point on the circle of radius 1 to a point on the circle of radius 3 if the segment cannot intersect the circle of radius 1.

______________ 14. **[7]** Find the value of $\frac{1}{3} + \frac{2}{9} + \frac{3}{27} + \frac{4}{81} + \frac{5}{243} + \ldots$.

______________ 15. **[7]** Bob is trying to type the word "welp". However, he has a $\frac{1}{8}$ chance of mistyping each letter and instead typing one of four adjacent keys, each with equal probability. Find the probability that he types "qelp" instead of "welp".

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______________ 16. **[7]** How many ways are there to tile a 2×12 board using an unlimited supply of 1×1 and 1×3 pieces?

______________ 17. **[7]** Jeffrey and Yiming independently choose a number between 0 and 1 uniformly at random. What is the probability that their two numbers can form the sidelengths of a triangle with longest side of length 1?

______________ 18. **[7]** On $\triangle ABC$ with $AB = 12$ and $AC = 16$, let M be the midpoint of BC and E, F be the points such that E is on AB, F is on AC, and $AE = 2AF$. Let G be the intersection of EF and AM. Compute $\frac{EG}{GF}$.

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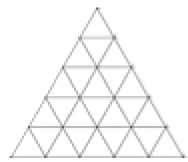
______________ 19. **[8]** Find the remainder when $2019x^{2019} - 2018x^{2018} + 2017x^{2017} - \cdots + x$ is divided by $x+1$.

______________ 20. **[8]** Parallelogram $ABCD$ has $AB = 5$, $BC = 3$, and $\angle ABC = 45°$. A line through C intersects AB at M and AD at N such that $\triangle BCM$ is isosceles. Determine the maximum possible area of $\triangle MAN$.

______________ 21. **[8]** Determine the number of convex hexagons whose sides only lie along the grid shown below.



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______________ 22. **[8]** Let $\triangle ABC$ be a triangle with side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Extend ray $\overrightarrow{AB}$ to a point D such that $AD = 12$, and similarly extend ray $\overrightarrow{CB}$ to point E such that $CE = 15$. Let M and N be points on the circumcircles of ABC and DBE, respectively, such that line MN is tangent to both circles. Determine the length of MN.

______________ 23. **[8]** A volcano will erupt with probability $\frac{1}{20-n}$ if it has not erupted in the last n years. Determine the expected number of years between consecutive eruptions.

______________ 24. **[8]** If x and y are integers such that $x + y = 9$ and $3x^2 + 4xy = 128$, find x.

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______________ 25. **[9]** Circle ω_1 has radius 1 and diameter AB. Let circle ω_2 be a circle with maximum radius such that it is tangent to AB and internally tangent to ω_1. A point C is then chosen such that ω_2 is the incircle of triangle ABC. Compute the area of ABC.

______________ 26. **[9]** Two particles lie at the origin of a Cartesian plane. Every second, the first particle moves from its initial position (x, y) to one of either $(x+1, y+2)$ or $(x-1, y-2)$, each with probability 0.5. Likewise, every second the second particle moves from it's initial position (x, y) to one of either $(x+2, y+3)$ or $(x-2, y-3)$, each with probability 0.5. Let d be the distance distance between the two particles after exactly one minute has elapsed. Find the expected value of d^2.

______________ 27. **[9]** Find the largest possible positive integer n such that for all positive integers k with $\gcd(k, n) = 1$, $k^2 - 1$ is a multiple of n.

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______ 28. **[11]** Let $\triangle ABC$ be a triangle with side lengths $AB = 13, BC = 14, CA = 15$. Let H be the orthcenter of $\triangle ABC$, M be the midpoint of segment BC, and F be the foot of altitude from C to AB. Let K be the point on line BC such that $\angle MHK = 90^\circ$. Let P be the intersection of HK and AB. Let Q be the intersection of circumcircle of $\triangle FPK$ and BC. Find the length of QK.

______ 29. **[11]** Real numbers (x, y, z) are chosen uniformly at random from the interval $[0, 2\pi]$. Find the probability that

$$\cos(x)\cdot\cos(y)+\cos(y)\cdot\cos(z)+\cos(z)\cdot\cos(x)+\sin(x)\cdot\sin(y)+\sin(y)\cdot\sin(z)+\sin(z)\cdot\sin(x)+1$$

is positive.

______ 30. **[11]** Find the number of positive integers where each digit is either 1, 3, or 4, and the sum of the digits is 22.

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______ 31. **[13]** In $\triangle ABC$, let D be the point on ray $\overrightarrow{CB}$ such that $AB = BD$ and let E be the point on ray $\overrightarrow{AC}$ such that $BC = CE$. Let L be the intersection of AD and circumcircle of $\triangle ABC$. The exterior angle bisector of $\angle C$ intersects AD at K and it follows that $AK = AB + BC + CA$. Given that points B, E, and L are collinear, find $\angle CAB$.

______ 32. **[13]** Let a be the largest root of the equation $x^3 - 3x^2 + 1 = 0$. Find the remainder when $\lfloor a^{2019} \rfloor$ is divided by 17.

______ 33. **[13]** For all $x, y \in \mathbb{Q}$, functions $f, g, h : \mathbb{Q} \to \mathbb{Q}$ satisfy $f(x + g(y)) = g(h(f(x))) + y$. If $f(6) = 2$, $g\left(\frac{1}{2}\right) = 2$, and $h\left(\frac{7}{2}\right) = 2$, find all possible values of $f(2019)$.

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________________ 34. **[15]** An *n-polyomino* is formed by joining n unit squares along their edges. A free polyomino is a polyomino considered up to congruence. That is, two free polyominos are the same if there is a combination of translations, rotations, and reflections that turns one into the other. Let $P(n)$ be the number of free n-polyominos. For example, $P(3) = 2$ and $P(4) = 5$. Estimate $P(20) + P(19)$. If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 10 \cdot \left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$

________________ 35. **[15]** Estimate

$$\sum_{k=1}^{2019} \sin(k),$$

where k is measured in radians. If your estimate is E and the actual value is A, your score for this problem will be

$$\max(0, 15 - 10 \cdot |E - A|).$$

________________ 36. **[15]** For a positive integer n, let $r_{10}(n)$ be the number of 10-tuples of (not necessarily positive) integers $(a_1, a_2, \ldots, a_9, a_{10})$ such that

$$a_1^2 + a_2^2 + \ldots + a_9^2 + a_{10}^2 = n.$$

Estimate $r_{10}(20) + r_{10}(19)$. If your estimate is E and the actual value is A, your score for this problem will be

$$\max\left(0, \left\lfloor 15 - 10 \cdot \left|\log_{10}\left(\frac{A}{E}\right)\right|\right\rfloor\right).$$